Switching controls

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Outline

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1. Motivation
2. Switching active controls
   - Motivation
   - The finite-dimensional case
   - The $1-d$ heat equation
   - Open problems
3. Flow control & Shocks
   - Motivation
   - Equation splitting
   - An example on inverse design
   - Open problems

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Motivation

- Systems with two ore more active controllers or design parameteres
- Systems with several components on the state (sometimes hidden !!!)

Goals

- Make control and optimization algorithms more performant by switching
- Develop strategies for switching

Related topics and methods

Splitting, domain decomposition, Lie's Theorem:

$$e^{A+B} = \lim_{n \to \infty} [e^{A/n} e^{B/n}]^n$$

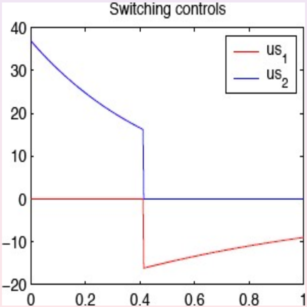
$$\varepsilon^{A+B} \sim e^{A/n} e^{B/n} \dots . e^{A/n} e^{B/n}, \quad \text{for } n \text{ large .}$$

Outline

Motivation

To develop systematic strategies allowing to build switching controllers.

The controllers of a system endowed with different actuators are said to be of switching form when only one of them is active in each instant of time.



Outline

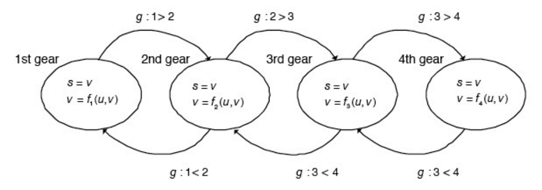
The finite-dimensional case

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = Ax(t) + u_1(t)b_1 + u_2(t)b_2 \\ x(0) = x^0. \end{cases} \tag{1}$$

$x(t) = \big(x_1(t), \ldots, x_N(t)\big) \in \mathbb{R}^N$ is the state of the system,
A is a $N \times N-$matrix,
$u_1 = u_1(t)$ and $u_2 = u_2(t)$ are two scalar controls
$b_1,\ b_2$ are given control vectors in $\mathbb{R}^N$.

More general and complex systems may also involve switching in the state equation itself:



$$x'(t) = A(t)x(t) + u_1(t)b_1 + u_2(t)b_2, \quad A(t) \in \{A_1, ..., A_M\}.$$

These systems are far more complex because of the nonlinear effect of the controls on the system.

Examples: automobiles, genetic regulatory networks, network congestion control,...

Controllability:

Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for control pairs (u_1, u_2) such that the solution of (1) satisfies

$$x(T) = x^1. \tag{2}$$

In the absence of constraints, controllability holds if and only if the Kalman rank condition is satisfied

$$\left[B, AB, \ldots, A^{N-1}B\right] = N \tag{3}$$

with $B = (b_1, b_2)$.

We look for **switching controls:**

$$u_1(t)u_2(t) = 0, \quad \text{a.e.} \quad t \in (0, T). \tag{4}$$

Under the rank condition above, these switching controls always exist.

The classical theory guarantees that the standard controls (u_1, u_2) may be built by minimizing the functional

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T \left[|b_1 \cdot \varphi(t)|^2 + |b_2 \cdot \varphi(t)|^2\right] dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),$$

among the solutions of the adjoint system

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), & t \in (0,\ T) \\ \varphi(T) = \varphi^0. \end{cases} \tag{5}$$

The rank condition for the pair (A, B) is equivalent to the following unique continuation property for the adjoint system which suffices to show the coercivity of the functional:

$$b_1 \cdot \varphi(t) = b_2 \cdot \varphi(t) = 0, \quad \forall t \in [0,\ T] \rightarrow \varphi \equiv 0.$$

The same argument allows considering, for a given partition $\tau = \{t_0 = 0 < t_1 < t_2 < ... < t_{2N} = T\}$ of the time interval $(0, T)$, a functional of the form

$$J_\tau\left(\varphi^0\right) = \frac{1}{2}\sum_{j=0}^{N-1}\int_{t_{2j}}^{t_{2j+1}} \left|b_1 \cdot \varphi(t)\right|^2 dt + \frac{1}{2}\sum_{j=0}^{N-1}\int_{t_{2j+1}}^{t_{2j+2}} \left|b_2 \cdot \varphi(t)\right|^2 dt$$

$$-x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0).$$

Under the same rank condition this functional is coercive too. In fact, in view of the time-analicity of solutions, the above unique continuation property implies the apparently stronger one:

$$b_1 \cdot \varphi(t) = 0 \quad t \in (t_{2j}, t_{2j+1});\ b_2 \cdot \varphi(t) = 0 \quad t \in (t_{2j+1}, t_{2j+2}) \rightarrow \varphi \equiv 0$$

and this one suffices to show the coercivity of J_τ. Thus, J_τ has an unique minimizer $\check{\varphi}$ and this yields the controls

$$u_1(t) = b_1 \cdot \check{\varphi}(t),\ t \in (t_{2j}, t_{2j+1}); \quad u_2(t) = b_2 \cdot \check{\varphi}(t),\ t \in (t_{2j+1}, t_{2j+2})$$

which are obviously of switching form.

Drawback of this approach:

- The partition has to be put a priori. Not automatic
- Controls depend on the partition
- Hard to balance the weight of both controllers. Not optimal.

Under further rank conditions, the following functional, which is a variant of the functional J, with the same coercivity properties, allows building switching controllers, without an a priori partition of the time interval $[0, T]$:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\left(\left|b_1\cdot\varphi(t)\right|^2, \left|b_2\cdot\varphi(t)\right|^2\right)dt - x^1\cdot\varphi^0 + x^0\cdot\varphi(0). \tag{6}$$

Theorem

Assume that the pairs $(A, b_2 - b_1)$ and $(A, b_2 + b_1)$ satisfy the rank condition. Then, for all $T > 0$, J_s achieves its minimum at least on a minimizer $\tilde{\varphi}^0$. Furthermore, the switching controllers

$$\begin{cases} u_1(t) = \tilde{\varphi}(t)\cdot b_1 & \text{when} \quad \left|\tilde{\varphi}(t)\cdot b_1\right| > \left|\tilde{\varphi}(t)\cdot b_2\right| \\ u_2(t) = \tilde{\varphi}(t)\cdot b_2 & \text{when} \quad \left|\tilde{\varphi}(t)\cdot b_2\right| > \left|\tilde{\varphi}(t)\cdot b_1\right| \end{cases} \tag{7}$$

where $\tilde{\varphi}$ is the solution of (5) with datum $\tilde{\varphi}^0$ at time $t = T$, control the system.

1. The rank condition on the pairs $\left(A,\ b_2 \pm b_1\right)$ is a necessary and sufficient condition for the controllability of the systems

$$x' + Ax = \left(b_2 \pm b_1\right) u(t). \tag{8}$$

This implies that the system with controllers b_1 and b_2 is controllable too but the reverse is not true.

2. The rank conditions on the pairs $\left(A,\ b_2 \pm b_1\right)$ are needed to ensure that the set

$$\left\{ t \in (0,\ T) : \ \left|\varphi(t) \cdot b_1\right| = \left|\varphi(t) \cdot b_2\right| \right\} \tag{9}$$

is of null measure, which ensures that the controls in (7) are genuinely of switching form.

Sketch of the proof:

There are two key points:
a) Showing that the functional J_s is coercive, i. e.,

$$\lim_{\|\varphi^0\|\to\infty} \frac{J_s(\varphi^0)}{\| \varphi^0 \|} = \infty,$$

which guarantees the existence of minimizers.
Coercivity is immediate since

$$|\varphi(t)\cdot b_1|^2 + |\varphi(t)\cdot b_2|^2 \leq 2\max\left[|\varphi(t)\cdot b_1|^2,\ |\varphi(t)\cdot b_2|^2\right]$$

and, consequently, the functional J_s is bounded below by a functional equivalent to the classical one J.
b) Showing that the controls obtained by minimization are of switching form.

This is equivalent to proving that the set

$$I = \{t \in (0,\ T) : |\tilde{\varphi} \cdot b_1| = |\tilde{\varphi} \cdot b_2|\}$$

is of null measure.

Assume for instance that the set $I_+ = \{t \in (0,\ T) : \tilde{\varphi}(t) \cdot (b_1 - b_2) = 0\}$ is of positive measure, $\tilde{\varphi}$ being the minimizer of J_s. The time analyticity of $\tilde{\varphi} \cdot (b_1 - b_2)$ implies that $I_+ = (0,\ T)$. Accordingly $\tilde{\varphi} \cdot (b_1 - b_2) \equiv 0$ and, consequently, taking into account that the pair $(A,\ b_1 - b_2)$ satisfies the Kalman rank condition, this implies that $\tilde{\varphi} \equiv 0$. This would imply that

$$J(\varphi^0) \geq 0,\ \forall \varphi^0 \in \mathbb{R}^N$$

which may only happen in the trivial situation in which $x^1 = e^{AT} x^0$, a trivial situation that we may exclude.

The Euler-Lagrange equations associated to the minimization of J_s take the form

$$\int_{S_1} \tilde{\varphi}(t)\cdot b_1\, \psi(t)\cdot b_1 dt + \int_{S_2} \tilde{\varphi}(t)\cdot b_2\, \psi(t)\cdot b_2 dt - x^1\cdot\psi^0 + x^0\cdot\psi(0) = 0,$$

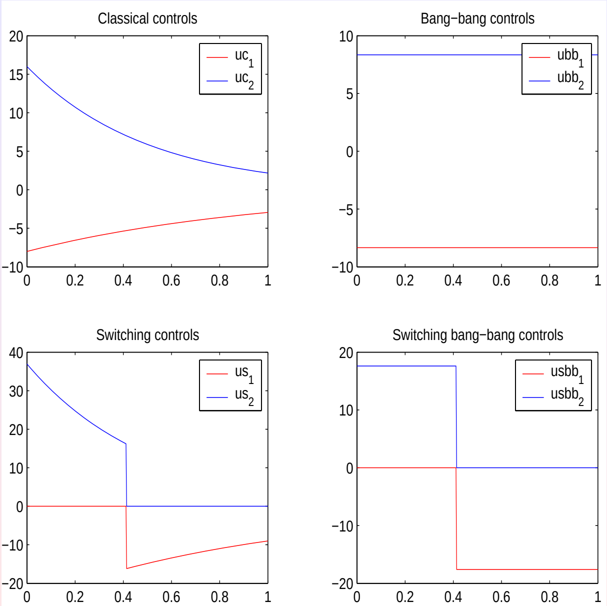
for all $\psi^0 \in \mathbb{R}^N$, where

$$\begin{cases} S_1 = \{t \in (0,\ T) : |\tilde{\varphi}(t)\cdot b_1| > |\tilde{\varphi}(t)\cdot b_2|\}, \\ S_2 = \{t \in (0,\ T) : |\tilde{\varphi}(t)\cdot b_1| < |\tilde{\varphi}(t)\cdot b_2|\}. \end{cases} \tag{10}$$

In view of this we conclude that

$$u_1(t) = \tilde{\varphi}(t)\cdot b_1\, 1_{S_1}(t), \quad u_2(t) = \tilde{\varphi}(t)\cdot b_2\, 1_{S_2}(t), \tag{11}$$

where 1_{S_1} and 1_{S_2} stand for the characteristic functions of the sets S_1 and S_2, are such that the switching condition holds and the corresponding solution satisfies the final control requirement.


Classical controls
uc1
uc2
Bang−bang controls
ubb1
ubb2
Switching controls
us1
us2
Switching bang−bang controls
usbb1
usbb2

Optimality:

The switching controls we obtain this way are of minimal $L^2\left(0,\ T;\ \mathbb{R}^2\right)$-norm, the space $\mathbb{R}^2$ being endowed with the ℓ^1 norm, i. e. with respect to the norm

$$||(u_1, u_2)||_{L^2(0,\ T;\, \ell^1)} = \left[\int_0^T (|\tilde{u}_1| + |\tilde{u}_2|)^2 dt\right]^{1/2}.$$

Outline

Failure of the switching strategy

Consider the heat equation in the space interval $(0, 1)$ with two controls located on the extremes $x = 0, 1$:

$$\begin{cases} y_t - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0, t) = u_0(t), \; y(1, t) = u_1(t), & 0 < t < T \\ y(x, 0) = y^0(x), & 0 < x < 1. \end{cases}$$

We look for controls $u_0, u_1 \in L^2(0, T)$ such that the solution satisfies

$$y(x, T) \equiv 0.$$

To build controls we consider the adjoint system

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ \varphi(0,\, t) = \varphi(1,\, t) = 0, & 0 < t < T \\ \varphi(x,\, T) = \varphi^0(x), & 0 < x < 1. \end{cases}$$

It is well known that the null control may be computed by minimizing the quadratic functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \left[|\varphi_x(0,\, t)|^2 + |\varphi_x(1,\, t)|^2\right] dt + \int_0^1 y^0(x)\varphi(x,\, 0)dx.$$

The controls obtained this way take the form

$$u_0(t) = -\hat{\varphi}_x(0,\, t);\; u_1(t) = \hat{\varphi}_x(1,\, t),\; t \in (0,\, T) \qquad (12)$$

where $\hat{\varphi}$ is the solution associated to the minimizer of J.

For building switching controls we rather consider

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\left[|\varphi_x(0,\, t)|^2, |\varphi_x(1,\, t)|^2\right] dt + \int_0^1 y^0(x)\varphi(x,\, 0)dx.$$

But for this to yield switching controls, the following UC is needed.
And it fails because of symmetry considerations!

$$\text{meas}\,\{t \in [0,\, T] : \varphi_x(0,\, t) = \pm\varphi_x(1,\, t)\} = 0.$$

This strategy yields switching controls for the control problem with two pointwise actuators:

$$\begin{cases} y_t - y_{xx} = u_a(t)\delta_a + u_b(t)\delta_b, & 0 < x < 1, \quad 0 < t < T \\ y(0,\, t) = y(1,\, t) = 0, & 0 < t < T \\ y(x,\, 0) = y^0(x), & 0 < x < 1, \end{cases}$$

under the irrationality condition

$$a \pm b \neq \frac{m}{k}, \forall k \geq 1,\, m \in \mathbb{Z}.$$

Outline

1. How many times do these controls switch?
2. In a general PDE setting this leads to unique continuation problems of the form:

$$\varphi_t + A^*\varphi = 0; |B_1^*\varphi| = |B_2^*\varphi| \to \varphi = 0??????$$

3. Systems where the state equation switches as well.

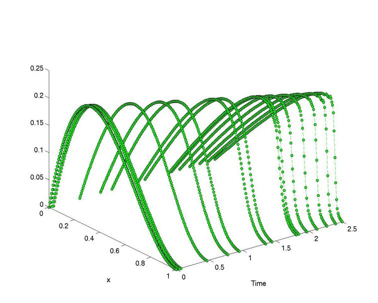
References:

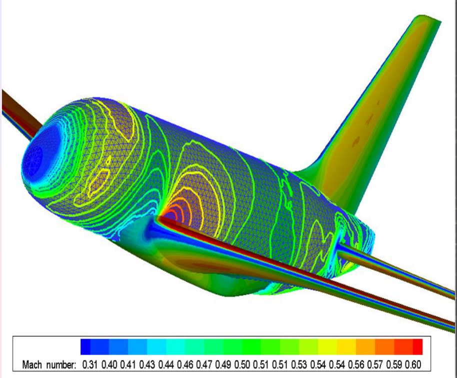
- M. Gugat, Optimal switching boundary control of a string to rest in finite time, preprint, October 2007.
- E. Z., Switching controls, preprint, 2008.

Outline

Solutions of hyperbolic systems may develop shocks or quasi-shock configurations and this may affect in a significant manner control and design problems.

- For shock solutions, classical calculus fails;
- For quasi-shock solutions the sensitivity is so large that classical sensitivity clalculus is meaningless.

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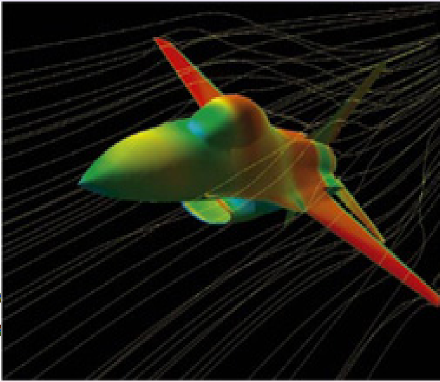


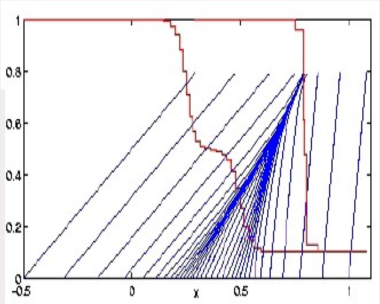
Burgers equation

- Viscous version:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0.$$

- Inviscid one:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$



Outline

In the inviscid case, the simple and "natural" rule

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \rightarrow \frac{\partial \delta u}{\partial t} + \delta u\frac{\partial u}{\partial x} + u\frac{\partial \delta u}{\partial x} = 0$$

breaks down in the presence of shocks

$\delta u =$ discontinuous, $\frac{\partial u}{\partial x} =$ Dirac delta $\Rightarrow \delta u \frac{\partial u}{\partial x}$????

The difficulty may be overcame with a suitable notion of measure valued weak solution using Volpert's definition of conservative products and duality theory (Bouchut-James, Godlewski-Raviart,...)

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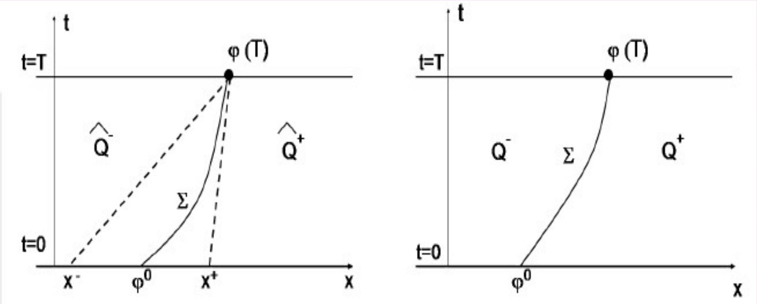
A new viewpoint: Solution = Solution + shock location. Then the pair (u, φ) solves:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, & \text{in } Q^- \cup Q^+, \\ \varphi'(t)[u]_{\varphi(t)} = \left[u^2/2\right]_{\varphi(t)}, & t \in (0, T), \\ \varphi(0) = \varphi^0, & \\ u(x,0) = u^0(x), & \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}. \end{cases}$$



The corresponding linearized system is:

$$\begin{cases} \partial_t \delta u + \partial_x (u \delta u) = 0, \quad \text{in } Q^- \cup Q^+, \\ \\ \delta\varphi'(t)[u]_{\varphi(t)} + \delta\varphi(t)\left(\varphi'(t)[u_x]_{\varphi(t)} - [u_x u]_{\varphi(t)}\right) \\ \quad + \varphi'(t)[\delta u]_{\varphi(t)} - [u \delta u]_{\varphi(t)} = 0, \quad \text{in } (0, T), \\ \\ \delta u(x, 0) = \delta u^0, \quad \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}, \\ \delta\varphi(0) = \delta\varphi^0, \end{cases}$$

Majda (1983), Bressan-Marson (1995), Godlewski-Raviart (1999), Bouchut-James (1998), Giles-Pierce (2001), Bardos-Pironneau (2002), Ulbrich (2003), ...
None seems to provide a clear-cut recipe about how to proceed within an optimization loop.

A new method

A new method: Splitting + alternating descent algorithm.

C. Castro, F. Palacios, E. Z., M3AS, 2008.

Ingredients:

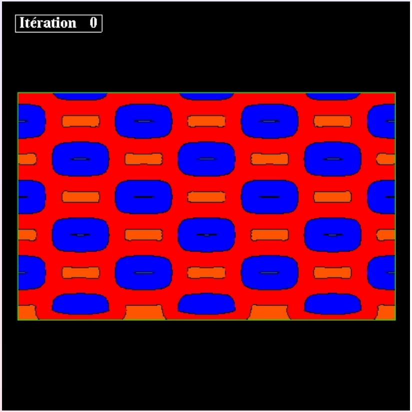
- The shock location is part of the state.

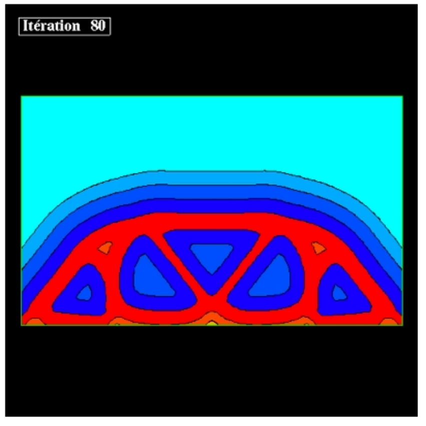
  State = Solution as a function + Geometric location of shocks.

- Alternate within the descent algorithm:
  - Shock location and smooth pieces of solutions should be treated differently;
  - When dealing with smooth pieces most methods provide similar results;
  - Shocks should be handeled by geometric tools, not only those based on the analytical solving of equations.

Lots to be done: Pattern detection, image processing, computational geometry,... to locate, deform shock locations,....

Compare with the use of shape and topological derivatives in elasticity:





Outline

An example: Inverse design of initial data

Consider

$$J(u^0) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

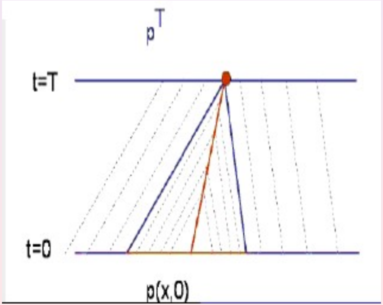
u^d = step function.
Gateaux derivative:

$$\delta J = \int_{\{x<\varphi^0\}\cup\{x>\varphi^0\}} p(x,0)\delta u^0(x)\, dx + q(0)[u]_{\varphi^0}\delta\varphi^0,$$

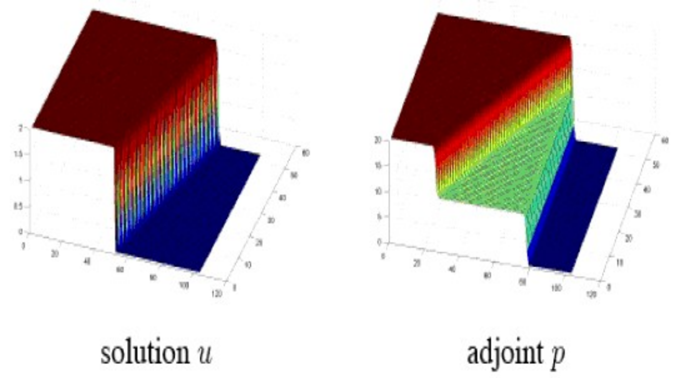
(p,q) = adjoint state

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } Q^- \cup Q^+, \\ [p]_\Sigma = 0, \\ q(t) = p(\varphi(t),t), \ \text{in } t \in (0,T) \\ q'(t) = 0, \ \text{in } t \in (0,T) \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x<\varphi(T)\}\cup\{x>\varphi(T)\} \\ q(T) = \dfrac{\frac{1}{2}\left[(u(x,T)-u^d)^2\right]_{\varphi(T)}}{[u]_{\varphi(T)}}. \end{cases}$$

- The gradient is twofold= variation of the profile + shock location.
- The adjoint system is the superposition of two systems = Linearized adjoint transport equation on both sides of the shock + Dirichlet boundary condition along the shock that propagates along characteristics and fills all the region not covered by the adjoint equations.



State u and adjoint state p when u develops a shock:



A new method: splitting+alternating descent

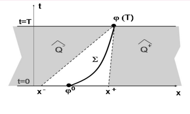
- Generalized tangent vectors $(\delta u^0, \delta\varphi^0) \in T_{u^0}$ s. t.

$$\delta\varphi^0 = \left(\int_{x^-}^{\varphi^0} \delta u^0 + \int_{\varphi^0}^{x^+} \delta u^0 \right) \Big/ [u]_{\varphi^0}.$$

do not move the shock $\delta\varphi(T) = 0$ and

$$\delta J = \int_{\{x<x^-\}\cup\{x>x^+\}} p(x,0)\delta u^0(x)\, dx,$$

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } \hat{Q}^- \cup \hat{Q}^+, \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\}. \end{cases}$$



For those descent directions the adjoint state can be computed by "any numerical scheme"!

- Analogously, if $\delta u^0 = 0$, the profile of the solution does not change, $\delta u(x,T) = 0$ and

$$\delta J = -\left[\frac{(u(x,T) - u^d(x))^2}{2}\right]_{\varphi(T)} \frac{[u^0]_{\varphi^0}}{[u(\cdot,T)]_{\varphi(T)}} \delta\varphi^0.$$

This formula indicates whether the descent shock variation is left or right!

WE PROPOSE AN ALTERNATING STRATEGY FOR DESCENT

In each iteration of the descent algorithm do two steps:

- Step 1: Use variations that only care about the shock location
- Step 2: Use variations that do not move the shock and only affect the shape away from it.

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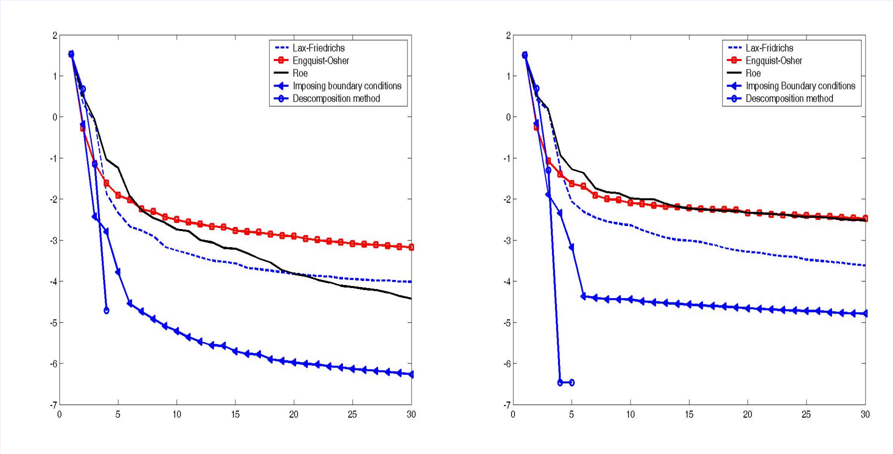
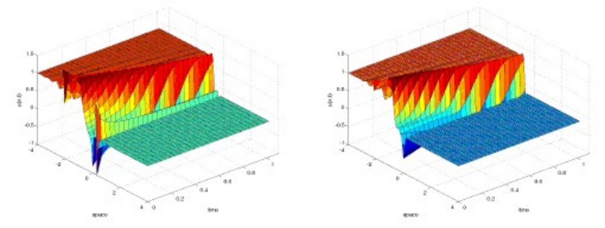
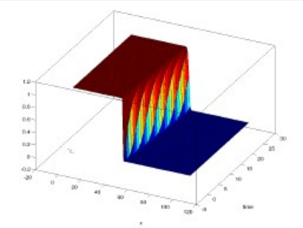


Splitting+Alternating wins!

Sol y sombra!



Results obtained applying Engquist-Osher's scheme and the one based on the complete adjoint system



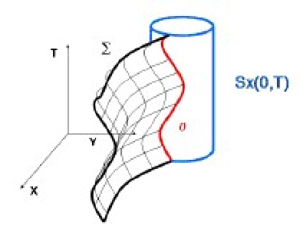
Splitting+Alternating method.

Splitting+alternating is more efficient:

- It is faster.
- It does not increase the complexity.
- Rather independent of the numerical scheme.

Extending these ideas and methods to more realistic multi-dimensional problems is a work in progress and much remains to be done.

Numerical schemes for PDE + shock detection + shape, shock deformation + mesh adaptation,...



Outline

Open problems

- More complex geometry of shocks
- Multi-dimensional problems: Shocks are located on hypersurfaces
- Adaptation to small viscosity: quasishocks
- Flux identification problems (F. James and M. Sepúlveda)
- Interpretation in the context of gradient methods: zig-zag gradient methods

$$z'(t) = -\nabla J(z); \quad \frac{z^{k+1} - z^k}{\Delta t} = -\nabla J(z^k).$$

Thank you!